Last time: Gaussian Elim

Solution Paradigms

A linear system has 3 possible solution paradigms:

- → No solutions ✗ (from an inconsistent equation)
- → Exactly 1 solution ✗
- → Infinitely many solutions. ← ✗

Thm: These are the only three possibilities...

Goal: Determine Solution sets.

Gave solutions as column vectors.

In general we give a full set of column vectors

Ex: Last time we solved

$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases} \rightsquigarrow \begin{cases} x = 0 \\ y = -1 + t \\ z = 5 - t \\ w = t \end{cases} \text{ for any } t \in \mathbb{R}$$

We write the solution set like so:

Solution set

$$\begin{bmatrix} 0 \\ -1+t \\ 5-t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \rightsquigarrow \left\{ \begin{bmatrix} 0 \\ -1 \\ 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

NB: this vector is a particular solution...

Matrices

A **matrix** is a rectangular array of numbers

Ex:
$$\begin{bmatrix} 0 & 1 \\ 1 & 5 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$
$$3 \times 2 \qquad\qquad 2 \times 2 \qquad\qquad 1 \times 3$$

An **$m \times n$ matrix** has m rows an n columns

A **column vector** is an $n \times 1$ matrix.
A **row vector** is a $1 \times n$ matrix.

The **entries** of a matrix are the numbers in the matrix
Entries are indexed by row and column.

Ex:
$$A = \begin{bmatrix} 0 & 1 & -1 & 2 & 5 \\ 1 & 0 & -3 & 0 & 2 \\ 0 & -7 & \pi & e & \gamma \end{bmatrix} \qquad a_{3,2} = -7$$

row number

column number

Convention: Matrices are represented w/ capital letters.
the corresponding entries are rep'd by the lowercase letter, so

$$D = [d_{i,j}].$$

We can represent a linear system via an **augmented matrix**.

Ex:
$$\begin{cases} 3x + 5y - 7z + w = 0 \\ \phantom{3x + {}} 5y - 3z + w = 5 \\ x \phantom{{}+5y} - z \phantom{{}+w} = 6 \end{cases} \rightsquigarrow \begin{array}{c} \begin{matrix} x & y & z & w & \end{matrix} \\ \left[\begin{array}{cccc|c} 3 & 5 & -7 & 1 & 0 \\ 0 & 5 & -3 & 1 & 5 \\ 1 & 0 & -1 & 0 & 6 \end{array}\right] \end{array}.$$

Let's solve this system w/ its matrix representation

NB: Gaussian elimination translates into "row operations" for the matrix set up.

Sol:
$$\left[\begin{array}{cccc|c} 3 & 5 & -7 & 1 & 0 \\ 0 & 5 & -3 & 1 & 5 \\ 1 & 0 & -1 & 0 & 6 \end{array}\right] \xrightarrow{\rho_3 \leftrightarrow \rho_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 6 \\ 0 & 5 & -3 & 1 & 5 \\ 3 & 5 & -7 & 1 & 0 \end{array}\right]$$

ρ = "rho"

$$\xrightarrow{\rho_3 - 3\rho_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 6 \\ 0 & 5 & -3 & 1 & 5 \\ 0 & 5 & -4 & 1 & -18 \end{array}\right] \xrightarrow{\rho_3 - \rho_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 6 \\ 0 & 5 & -3 & 1 & 5 \\ 0 & 0 & -1 & 0 & -23 \end{array}\right]$$

$$\xrightarrow[-\rho_3]{\frac{1}{5}\rho_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 6 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & 23 \end{array}\right] \xrightarrow[\rho_2 + \frac{3}{5}\rho_3]{\rho_1 + \rho_3} \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 29 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{74}{5} \\ 0 & 0 & 1 & 0 & 23 \end{array}\right]}_{\text{"Reduced Row Echelon Form"}}$$

"first nonzero entry of each row is a 1 and sees only 0's above and below" → "Reduced Row Echelon Form"

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 29 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{74}{5} \\ 0 & 0 & 1 & 0 & 23 \end{array}\right] \rightsquigarrow \begin{cases} x = 29 \\ y + \frac{1}{5}w = \frac{74}{5} \\ z = 23 \end{cases}$$

$$\rightsquigarrow \begin{cases} x = 29 \\ y = \frac{74}{5} - \frac{1}{5}t \\ z = 23 \\ w = t \end{cases} \text{ for } t \in \mathbb{R} \qquad \text{OR} \quad \begin{cases} x = 29 \\ y = s \\ z = 23 \\ w = 74 - 5s \end{cases} \text{ for } s \in \mathbb{R}$$

Hence we have solution set
$$\left\{ \begin{bmatrix} 29 \\ \frac{74}{5} \\ 23 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

OR
$$\left\{ \begin{bmatrix} 29 \\ 0 \\ 23 \\ 74 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ -5 \end{bmatrix} : s \in \mathbb{R} \right\}$$

Same solution set, different form!

$\boxed{/\!/}$

Ex: Solve $\begin{cases} x_1 + x_3 = 4 \\ x_1 - x_2 + 2x_3 = 5 \\ 4x_1 - x_2 + 5x_3 = 17 \end{cases}$.

Sol: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right] \xrightarrow[\ell_3 - 4\ell_1]{\ell_2 - \ell_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[\ell_3 + \ell_2]{-\ell_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore \begin{cases} x_1 + x_3 = 4 \\ x_2 - x_3 = -1 \end{cases} \rightsquigarrow \begin{cases} x_1 = 4 - t \\ x_2 = -1 + t \\ x_3 = t \end{cases}$

$\therefore$ Solution set is $\left\{ \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$ $\boxed{//}$

$\therefore$ = "therefore"

Ex: Solve $\begin{cases} 3x + 2y = 5 \\ -6x - 4y = 0 \end{cases}$

Sol: $\left[\begin{array}{cc|c} 3 & 2 & 5 \\ -6 & -4 & 0 \end{array}\right] \xrightarrow{\ell_2 + 2\ell_1} \left[\begin{array}{cc|c} 3 & 2 & 5 \\ 0 & 0 & 10 \end{array}\right] \leftarrow$

$\therefore$ $0 = 10$ is implied by the second row,

So the solution set is $\emptyset = \{\}$ $\boxed{//}$

$\uparrow$ empty set.

Preview of Coming Attractions: Matrix Algebra.

Operations on matrices (today):

→ Normal row operations
(swap, add, multiply).

→ $\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \underset{\text{matrix addition}}{+} \underbrace{t \begin{bmatrix} 7 \\ 0 \\ 5 \end{bmatrix}}_{\text{scalar multiplication}} = \begin{bmatrix} 7t \\ 1 \\ 5+5t \end{bmatrix}$

<u>Defⁿ</u>: Let A and B be $m \times n$ matrices and let $c \in \mathbb{R}$ be constant.
The <u>sum</u> of A and B is $A+B = [a_{i,j} + b_{i,j}]$,
i.e. the matrix obtained by entry-wise addition.
The <u>scalar multiple</u> of A by c is $cA = [c a_{ij}]$,
i.e. the matrix obtained from multiplying each entry of A by c.

<u>Ex</u>: $\underset{2\times 3}{\begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} + \begin{bmatrix} 7 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 3+7 & -1-1 & 0+0 \\ 2+0 & 0-1 & 1-1 \end{bmatrix} = \begin{bmatrix} 10 & -2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$

$5 \begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 5\cdot 2 & 5\cdot 3 \\ 5\cdot 1 & 5\cdot -3 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 5 & -15 \end{bmatrix}$

<u>Non-ex</u>: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 7 & -2 \end{bmatrix}$ IS UNDEFINED!